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Solving restricted line location problems via a dual interpretation

Anita Schöbel *

Fachbereich Mathematik, Universität Kaiserslautern, Postfach 3049, D-67653 Kaiserslautern, Germany

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Abstract

In line location problems the objective is to find a straight line which minimizes the sum of distances, or the maximum distance, respectively, to a given set of existing facilities in the plane. These problems have been well solved. In this paper we deal with restricted line location problems, i.e. we have given a set in the plane where the line is not allowed to pass through. With the help of a geometric duality we solve such problems for the vertical distance and then extend these results to block norms and some of them even to arbitrary norms. For all norms we give a finite candidate set for the optimal line. © 1999 Elsevier Science B.V. All rights reserved.

1. Introduction

The problem of locating a straight line in the plane to approximate a given point set is well known in location theory, statistics, and in computational geometry and has applications in all three disciplines (see e.g. the surveys of [9,10,17,19]). Given a set $\mathscr{E}x = \{A_1, A_2, \ldots, A_M\}$ of existing facilities represented by points in the plane, we are looking for a straight line minimizing the sum of weighted distances to the existing facilities, or the maximum weighted distance to the existing facilities, respectively. This problem has been well solved for various distance functions d (measuring the distance $d(A_m, l) = \min_{P \in l} d(A_m, P)$ between an existing facility and a line). For the rectangular distance $d = l_1$, [13,16,24] give efficient solution approaches, and for $d = l_2$ is the Euclidean distance, the problem is studied among others in [8,11,13,15]. With block norms [20] gives an efficient algorithm and general norms are discussed in [22]. Practical applications of line location problems include the planning of a new highway or a railway close to some given cities, or the construction of conveyor belts, or drainage- and irrigation ditches, see [15].

If, however, a forbidden region R is introduced, where the line is not allowed to pass through, e.g. R can be a lake, or a natural habitat, or some industrial area, we have a

* Fax: +49-631-29081; e-mail: schoebel@mathematik.uni-kl.de.

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restricted location problem. In classical facility location problems with restricted sets have been discussed by e.g. [4], but for line location this problem has not been studied so far. (Problems where the line is forced to pass through one given point have been discussed by [16].)

In this paper we formulate such restricted line location problems and give some structural results and algorithmic approaches. In our discussion we do not restrict ourselves to some special norms, but most of our results are true, if the distance measure is derived from any norm.

The paper is organized in the following way. In the next section we give a formal definition of restricted line location problems and we also repeat some known results for the unrestricted case and introduce some results about piecewise linear problems. Section 3 introduces a geometric duality and solves restricted line location problems for the vertical distance, both for the sum and the maximum objective function. In Sections 4 and 5 the results of Section 3 will be generalized first to block norms and then to all distances derived from norms. Possible extensions are given in Section 6.

2. Problem description and basic concepts

2.1. Locating lines in the plane

Formally, the problem of locating a line in the plane can be stated as follows. Given an index set $\mathcal{M} = \{1, 2, \dots, M\}$ and for all $m \in \mathcal{M}$ an existing facility $A_m = (a_{m1}, a_{m2}) \in \mathbb{R}^2$ with nonnegative weight $w_m \geqslant 0$, find a line l such that

$$f(l) = \sum_{m \in \mathcal{M}} w_m d(A_m, l)$$

is minimized (then l is called a median line) or such that

$$g(l) = \max_{m \in \mathcal{M}} w_m d(A_m, l)$$

is minimized, respectively (then l is called a center line). The set of optimal lines is usually denoted by $\mathcal{L}^*$. Here

$$d(A, l) = \min_{P \in l} d(A, P)$$

gives the distance between any point $A \in \mathbb{R}^2$ and a straight line $l \subseteq \mathbb{R}^2$. In the following we will use the five-position classification scheme which has been developed in [5,6]. The problems described above are in this scheme classified as $1l/\mathbb{R}^2/\cdot/d/\sum$ and $1l/\mathbb{R}^2/\cdot/d/\max$, respectively, meaning in short, that we want to locate one line ($1l$) in the plane ($\mathbb{R}^2$) with no special assumptions ($\cdot$) using the distance measure d and minimizing the sum of distances f between the existing facilities and the line ($\sum$), or the maximum distance g (max), respectively.

As mentioned in the introduction these problems have been well solved. The main result for the median problem $1l/\mathbb{R}^2/\cdot/d/\sum$ is that there always exists a line passing through at least two of the existing facilities. This was first proved for Euclidean and rectangular distances by [13,15] and recently shown by [22] for all distances derived from norms. For $d = l_2$, the Euclidean distance, [9] (see also [9]) showed the sharper result that *all* optimal lines are passing through at least two of the existing facilities. This is not true for all norms, see the counterexample given in [12]. For the center problem with an arbitrary norm γ as distance measure $(1l/\mathbb{R}^2/\cdot/\gamma/\max)$, there exists an optimal line which is at maximum distance from at least three of the existing facilities, see also [22]. As we need to refer to that result later on, we formulate it as our first theorem.

Theorem 1. *For all distances derived from norms the following holds.*
- For the median problem *there exists an optimal line passing through two of the existing facilities.*
- For the center problem *there exists an optimal line which is at maximum distance from three of the existing facilities.*

Now suppose there is an area in the plane (a restricted set R) where no line is allowed to pass through. Then the two restricted line location problems can be written as

$$\begin{array}{llcll} \min & f(l) & \quad or \quad & \min & g(l) \\ \text{s.t.} & l \cap int(R) = \emptyset & & \text{s.t.} & l \cap int(R) = \emptyset, \text{ respectively.} \end{array}$$

In the classification scheme these restricted problems are given by $1l/\mathbb{R}^2/R/d/\sum$ and $1l/\mathbb{R}^2/R/d/\max$, respectively.

The following notation will also be used throughout the paper:

$$l_{s,b} = \{X = (x_1, x_2) \in \mathbb{R}^2 \colon x_2 = sx_1 + b\}$$

denotes a non-vertical line with slope s and intercept b. For a set $R \subseteq \mathbb{R}^2$ let ∂R denote the boundary of R, $int(R)$ the interior of R, $conv(R)$ denotes the convex hull, and $ext(R)$ the set of extreme points of R, which may be empty.

2.2. Piecewise linear convex problems

For solving restricted line location problems we will use the theory of piecewise linear convex problems with restrictions developed in [5] for classical location problems and extended in [18] to general piecewise linear convex problems. Suppose we have given a set of lines $\mathscr{H} = \{h_1, h_1, \ldots, h_n\}$ which partitions the plane into cells ($\mathscr{H}$ is called the set of construction lines) and a convex function f which is linear on each cell. To minimize f the theory of linear programming shows that there always exists an extreme point v of a cell which is optimal. Introducing a restricted set R, the problem to consider now is

$$(ROL) \quad \min f(x) \quad \text{s.t. } x \notin int(R).$$

Then one can use the geometric properties of the level sets $L_{\leqslant}(t) = \{x\colon f(x) \leqslant t\}$ and level curves $L_{=}(t) = \{x\colon f(x) = t\}$ of f to show the following three results, which will be needed in the next sections. Let $\mathscr{X}^*$ denote the set of all optimal solutions of the unrestricted problem and $\mathscr{X}_R^*$ the set of all optimal solutions of (ROL). Since f is convex, the following result holds.

Theorem 2. *If $\mathscr{X}^* \subseteq int(R)$ we have $\mathscr{X}_R^* \subseteq \partial R$, i.e. all optimal solutions of the restricted problem are contained in the boundary of R.*

For convex sets we also know the following (see Theorem 6 in [18]).

Theorem 3. *Let $R \subseteq \mathbb{R}^2$ be convex and $\mathscr{X}^* \subseteq int(R)$. Then there exists an optimal solution $x_R^* \in \mathscr{X}_R^*$ such that x_R^* is a zero-dimensional intersection between the boundary ∂R and a construction line $h \in \mathscr{K}$, i.e. there exists an optimal solution x_R^* in the finite set of points*

$$Cand = \{h \cap \partial R\colon h \in \mathscr{K} \text{ and } dim(h \cap \partial R) = 0\}.$$

If R is not convex, but a simple polygon, we use the following result (see Lemma 8 in [18]).

Theorem 4. *Let the restricted set R be a simple polygon and let $\mathscr{X}^* \subseteq int(R)$. Then there exists an optimal solution x_R^* such that $x_R^* \in Cand_{polygon}$, where*

$$Cand_{polygon} = Cand \cup \{x\colon x \text{ is a reflexive vertex of } R\}.$$

3. A geometric duality to solve restricted problems with vertical distance

In this section we are concerned with the vertical distance d_{ver}. The vertical distance between a point $A = (a_1, a_2)$ and a non-vertical line $l = l_{s,b}$ is given by the length of the vertical line segment between A and l and can be calculated by

$$d_{ver}(A, l) = |a_1 s - a_2 + b|.$$

If l is a vertical line, then $d_{ver}(A, l) = \infty$, meaning that a vertical line can never be optimal unless all existing facilities have the same first coordiante $a_{m1} = a_1$ for all $m \in \mathscr{M}$, but that case is trivial and will therefore be neglected. Summarizing this, the objective function of the median and the center problem $(1l/\mathbb{R}^2/\cdot/d_{ver}/\cdot)$ is given by

$$f(l_{s,b}) = \sum_{m \in \mathscr{M}} w_m |a_{m1} s - a_{m2} + b|$$

and

$$g(l_{s,b}) = \max_{m \in \mathscr{M}} w_m |a_{m1} s - a_{m2} + b|,$$

respectively.

Note that both functions are convex in the two variables s, b.

Now consider the following transformation T (already introduced in [21]) mapping points to non-vertical lines and vice versa. Let $A = (a_1, a_2)$ be a point and $l_{s,b}$ a non-vertical line.

$$T(A) := l_{-a_1,a_2} = \{(s,b) \colon b = -a_1 s + a_2\},$$
$$T(l_{s,b}) := (s,b).$$

The space of the transformed points and lines will be called the dual space throughout that paper. It can easily be checked that the transformation keeps the vertical distance between points and lines, as the following lemma states.

Lemma 1. *Let A be a point and l be a line. Then we have*

$$d_{ver}(A,l) = d_{ver}(T(l), T(A)),$$

especially we have $A \in l \Leftrightarrow T(l) \in T(A)$.

Therefore we conclude the following theorem.

Theorem 5. *The problem of locating a line minimizing the sum (the maximum) of weighted vertical distances to a given set of points $\{A_1, A_2, \ldots, A_M\}$ is equivalent to the problem of locating a point minimizing the sum (the maximum) of vertical distances to a given set of lines $\{T(A_1), T(A_2), \ldots, T(A_M)\}$, i.e. $1l/\mathbb{R}^2/\cdot/d_{ver}/\sum$ is equivalent to $1/\mathbb{R}^2/\mathscr{E}x = \{T(A_1), \ldots, T(A_M)\}/d_{ver}/\sum$ and $1l/\mathbb{R}^2/\cdot/d_{ver}/\max$ is equivalent to $1/\mathbb{R}^2/\mathscr{E}x = \{T(A_1), \ldots, T(A_M)\}/d_{ver}/\max$.*

Other transformations mapping points to lines and lines to points are often used in projective geometry, similar transformations to the one given above which also transform points to lines and vice versa and which are keeping the distances can be found in [1,2,7,23].

Consider a location problem with the following five existing facilities $A_1 = (1, -1)$, $A_2 = (-1, 1)$, $A_3 = (-1, 2)$, $A_4 = (0, 1)$, and $A_5 = (-\frac{1}{2}, -1)$. Then Fig. 1 *shows the set of existing facilities and the unique optimal lines l^*_{med} for the median problem and l^*_{cen} for the center problem, respectively (which are parallel in this example). The transformed existing facilities $L_m = T(A_m)$, $m = 1, \ldots, 5$ and the optimal solutions X^*_{med} and X^*_{cen} in dual space are shown in* Figs. 2 *and* 5.

3.1. The median problem

Consider the set of lines $\mathscr{K}^{med} := \{L_m = T(A_m) \colon m \in \mathscr{M}\}$ which partitions the dual $\mathbb{R}^2$ into cells, see Fig. 2. On each cell, for all $m \in \mathscr{M}$, no sign of $(a_{m1}s - a_{m2} + b)$ changes such that the objective function f of $1l/\mathbb{R}^2/\cdot/d_{ver}/\sum$ is linear on each cell. As f also is convex we have a piecewise linear convex problem (see Section 2.2). We therefore know that there exists an optimal solution which is an extreme point V of a cell. As

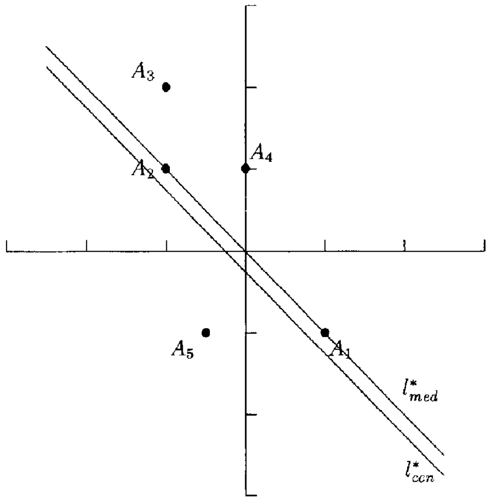

Fig. 1. An example with five existing facilities and the unique optimal solutions for the median and the center problem.

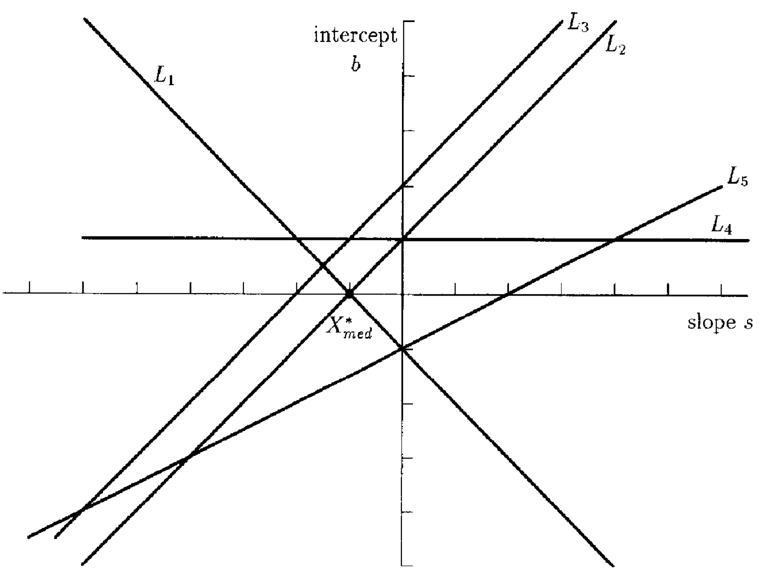

Fig. 2. Construction lines and optimal solution for the median problem in dual space.

all cell vertices lie on at least two different lines $T(A_m), T(A_k) \in \mathscr{H}^{med}$ we conclude that the line $T^{-1}(V)$ in the original space passes through the two points A_m and A_k – a short proof for the fact that there always exists an optimal line for $1l/\mathbb{R}^2/\cdot/d_{ver}/\sum$ passing through at least two of the existing facilities.

In dual space we can use the theory introduced in Section 2.2 to solve a large class of restricted problems. The following result follows immediately from Theorem 3.

Theorem 6. *Let R^{T} be a convex forbidden set in dual space. Then there exists an optimal solution X_R^* in dual space, such that*
- *either X_R^* also is an optimal solution for the unrestricted problem,*
- *or*

$$X_R^* \in Cand = \{X \colon X \in \partial R^{\mathrm{T}} \cap T(A_m) \ for\ one\ m \in \mathcal{M}\},$$

where no one-dimensional intersections have to be considered.

For convex forbidden sets R^{T} this means that we only have to investigate the intersection points between all lines $T(A_m) \in \mathscr{K}$ with the boundary of the restricted set R^{T}, yielding an efficient geometric approach to solve the restricted problem.

To solve the restricted line location problem in primal space we now proceed as follows. We transform the original problem and the restricted set R to dual space, where

$$R^{\mathrm{T}} := T(R) = \{X \colon T^{-1}(X) \cap R \neq \emptyset\}$$

is the set of all points in dual space corresponding to lines which intersect the forbidden region R in the original space. For this transformation of R to dual space, we have the following easy property, already mentioned in [7,23] for a similar dual transformation.

Lemma 2. *Let R be convex. Then $X \in \partial R^{\mathrm{T}}$ if and only if $T^{-1}(X)$ touches R.*

Two more properties are necessary.

Lemma 3. *Let $R \subseteq \mathbb{R}^2$. Then we have the following:*

1. *If R is connected, then $T(R) = T(conv(R))$.*
2. *$T(R)$ is convex if and only if $|R| = 1$ or there is a vertical line contained in $conv(R)$ or R is a vertical line segment.*

Proof. (1) This follows from the fact, that a line l meets a connected set R if and only if l meets $conv(R)$, see e.g. [23].

(2) If R consists only of one point P then $T(R) = T(P)$ is a (convex) line and if all non-vertical lines intersect R then $T(R)$ is the whole dual space. If R is a vertical line segment with endpoints X and Y we have that $T(R)$ is the (convex) strip between the parallel lines $T(X)$ and $T(Y)$.

For the other direction, first suppose there exist two points $X = (x_1, x_2)$, $Y = (y_1, y_2) \in R$ with $x_1 < y_1$. Now take any non-vertical line $l = l_{s,b}$ not intersecting R and choose a point $P = (p_1, p_2) \in l$ with $x_1 < p_1 < y_1$. Consider the two lines l_1 through X and P and l_2 through Y and P. For the slopes s_1 and s_2 of these lines we have $s_1 > 0$ and $s_2 < 0$. All three lines intersect in P, i.e. $T(l), T(l_1)$, and $T(l_2)$ all lie on the line $T(P)$

(see Lemma 1) and furthermore $s_2 < s < s_1$, such that l is a convex combination of l_1 and l_2. As $T(l) \notin T(R)$, but $T(l_1), T(l_2) \in R$ we have that R is not convex.

Now suppose that R is contained in a vertical line. As R is not a line segment, we find two points $X_1 = (x, b_1)$, $X_2 = (x, b_2) \in R$ and some point $Y = (x, b) \notin R$ in between X_1 and X_2, i.e. without restriction $b_1 < b < b_2$. That means, the horizontal line through Y is a convex combination between the horizontal lines through X_1 and X_2, but does not intersect R. □

With Lemma 2, our original problem is equivalent to the following problem in dual space:

$$\min f(X) \quad \text{s.t.}\ X \notin int(R^{\mathrm{T}}),$$

which is a version of (ROL) and is therefore easily solvable for convex sets R^{T}. Unfortunately, according to Lemma 3 we have that for all two-dimensional sets R the transformed set R^{T} never is convex, if there is any feasible line for the original problem, such that a simple enumeration of a candidate set as mentioned in Theorem 6 does not solve the restricted line location problem. But we can conclude the following:

Theorem 7. *If no optimal line for the unrestricted problem is feasible for the restricted problem then any line solving the restricted problem* $1l/\mathbb{R}^2/R/d_{ver}/\sum$ *is a tangent to R.*

Proof. In dual space we conclude from Theorem 2 that all optimal solutions lie on the boundary of $T(R)$. If R is convex, we directly apply Lemma 2 and get the result. If R is not convex, we look at $conv(R)$ according to Lemma 3 and get an optimal solution, which is a tangent to $conv(R)$, and therefore also to R. □

For arbitrary restricted sets R there are infinitely many tangents which have to be considered to solve the restricted problem. For polygone sets, however, there exists a finite candidate set for the optimal solution of the restricted problem. For this we need the following lemma, which is illustrated in Fig. 3.

Lemma 4. *Let R be a simple polygon. Then* $T(R)$ *is a non-convex (non-finite) polygon in dual space and the following hold:*
1. (s, b) *is a vertex of* $T(R)$ *if and only if* $l_{s,b}$ *contains a non-vertical facet of* $conv(R)$.
2. V *is a vertex of* $conv(R)$ *if and only if* $T(V)$ *contains a non-vertical facet of* $T(R)$.

Proof. Using Lemma 2 we only have to check the tangents of R. Exactly those dual points corresponding to tangents passing through a vertex V of R lie on the line $T(V)$ (see Lemma 1) and therefore form an edge of $T(R)$, and as each facet of R contains two vertices we conclude that exactly those lines l containing an edge of $conv(R)$ correspond to points on two edges of $T(R)$, i.e. to the vertices of $T(R)$. □

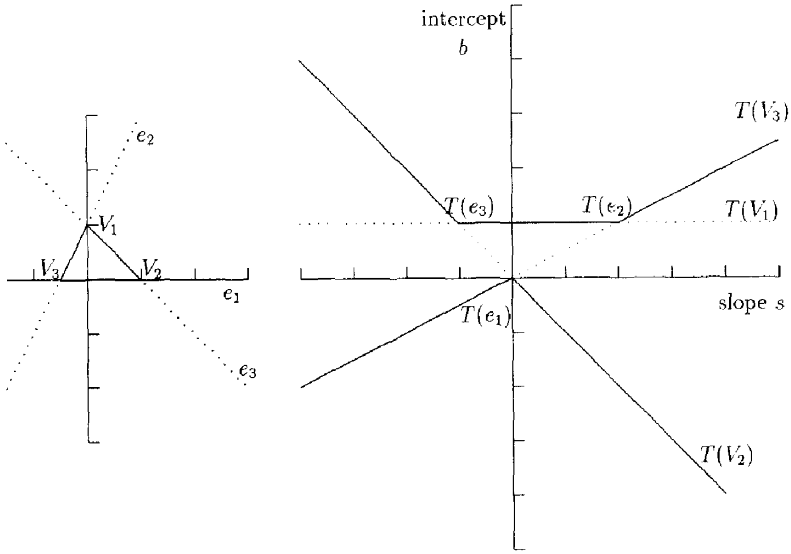

Fig. 3. Transformation of a triangle to dual space.

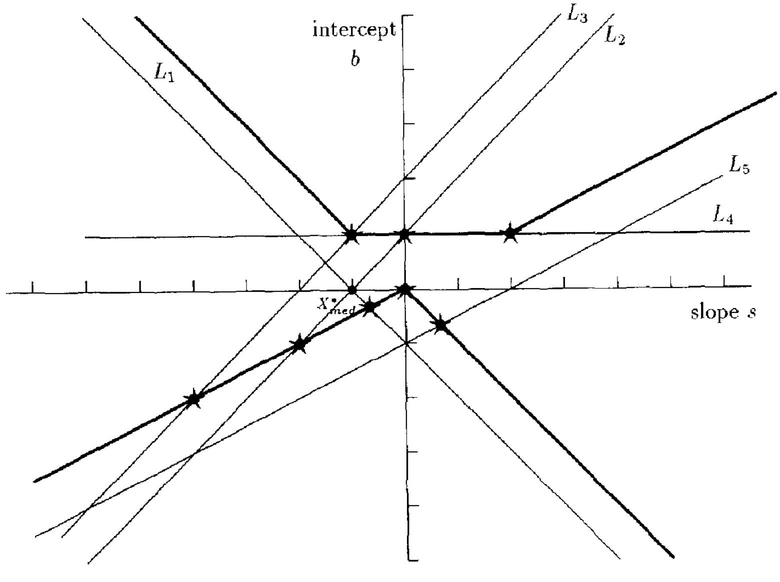

Fig. 4. The restricted set in dual space. The candidate points are marked by stars.

The situation of the following theorem in dual space is illustrated in Fig. 4.

Theorem 8. *Let R be a simple polygon. For the restricted problem* $1l/\mathbb{R}^2/R = Polygon/d_{ver}/\sum$ *we have*

- *Either an optimal line for the unrestricted problem is feasible,*
- *or there exists an optimal line for the restricted problem which is a facet of R or which passes through one of the existing facilities and a vertex of R, i.e. there exists a line $l \in Cand_{Poly}$ where*

$$Cand_{Poly} := \{\textit{lines } l\text{: } l \textit{ is a facet of } R\} \\ \cup \{\textit{lines } l\text{: there exist } m \in \mathcal{M}, \mathcal{V} \in ext(R)\text{: } A_m, V \in l\}.$$

Proof. Using Lemma 3 we transform $conv(R)$ to dual space and apply Theorem 4. Therefore we know that there exists an optimal solution X, which is

- either an intersection point between a construction line $T(A_m)$ and $\partial T(R)$, in this case the line $T^{-1}(X)$ touches ∂R (see Lemma 2) and contains A_m (see Lemma 1)
- or an inner vertex of $T(R)$, in this case the line $T^{-1}(X)$ is a facet of $conv(R)$ (see Lemma 4). □

Note that for d_{ver} it is also possible to calculate the set of all optimal solutions of the restricted problem $\mathcal{X}_R^*$ by the following formula. If $t_R^* = f(l_R^*)$ denotes the objective value of the restricted problem then the set of optimal solutions in dual space is given by the intersection of the level set $L_{\leqslant}(t_R^*)$ and the boundary of the transformed restricted set, i.e. $\mathcal{X}_R^* = L_{\leqslant}(t_R^*) \cap \partial T(R)$, which corresponds to a set of tangents in primal space.

3.2. The unweighted center problem

For the center problem we use the same theory as for the median problem as g also is piecewise linear and convex. Only the cell structure differs. Let us call U and L the upper and the lower envelope of the set of lines $\{T(A_1), \dots, T(A_M)\}$ and define the mid-line as

$$h^{Mid} = \{X\text{: } d_{ver}(X, U) = d_{ver}(X, L) = g(T^{-1}(X))\}.$$

Note that h^{Mid} is piecewise linear with breakpoints whose first coordinates coincide with the first coordinates of the breakpoints of U and L. Let us furthermore denote by $\mathcal{H}_L$ the set of all first coordinates of breakpoints of L and analogously let $\mathcal{H}_U$ be the set of first coordinates of breakpoints of U. Then we define the following two sets of halflines:

$$h_z^l = \{X = (z, x_2)\text{: } X \text{ lies above } h_{Mid}\} \quad \text{for all } z \in \mathcal{H}_L,$$
$$h_z^u = \{X = (z, x_2)\text{: } X \text{ lies below } h_{Mid}\} \quad \text{for all } z \in \mathcal{H}_U.$$

We now define the construction lines for the unweighted center problem as

$$\mathcal{K}^{cen} = \{h^{Mid}, h_{z_1}^l, h_{z_2}^u, z_1 \in \mathcal{H}_L, z_2 \in \mathcal{H}_u\}.$$

Notice that g is linear on the cells which are defined by them (see Fig. 5).

Again we transform the restricted center problem to dual space and then know from Lemma 2, that it is equivalent to

$$\min g(X) \quad \text{s.t. } X \notin int(R^{\mathrm{T}}),$$

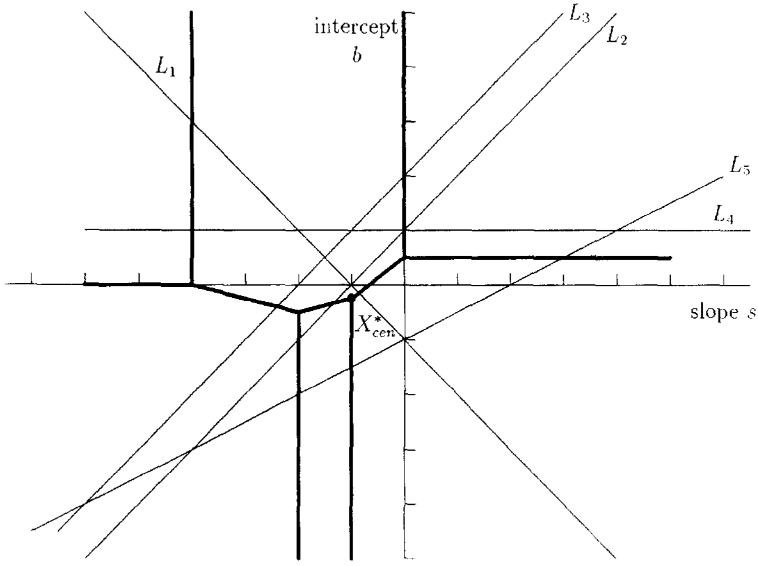

Fig. 5. Construction lines and optimal solution for the center problem in dual space.

where the objective $g(X)$ is interpreted as the maximum distance from point X to the set of lines $\{T(A_m)$: $m \in \mathcal{M}\}$. For solving the unrestricted problem we again know that there exists an optimal solution V which is a vertex of a cell, that means in our case $V \in h^{Mid}$ and the corresponding line $T^{-1}(V)$ is at maximum distance from at least three existing facilities according to Lemma 1. For the following we also note that for all points $X \in h$ for any $h \in \mathcal{H}^{cen}$ we have that the line $T^{-1}(X)$ is at maximum distance from at least two of the existing facilities. From this fact and from Theorems 2–4, and Lemmas 2–4 we conclude – as for the median problem – our next results.

Theorem 9. *Let R^{T} be a convex forbidden set in dual space. Then there exists an optimal solution X^*_R in dual space, such that*

- *either $T^{-1}(X^*_R)$ also is an optimal solution for the unrestricted problem,*
- *or*

$$X^*_R \in Cand = \{X\colon X \in \partial R^{\mathrm{T}} \cap h \text{ for one } h \in \mathcal{H}^{cen}\},$$

where no one-dimensional intersections have to be considered.

Theorem 10. *If no optimal line for the unrestricted problem is feasible for the restricted problem then any line solving the restricted unweighted problem $1l/\mathbb{R}^2/R$, $w_m = 1/d_{ver}/\max$ is a tangent to R.*

Theorem 11. *Let R be a simple polygon. For the restricted problem $1l/\mathbb{R}^2/R =$ Polygon, $w_m = 1/d_{ver}/\max$ we then have:*

- *Either an optimal line for the unrestricted problem is feasible*
- *or there exists an optimal line for the restricted problem which is a facet of R or which passes through a vertex of R and is at maximum distance from two of the existing facilities, i.e. there exists a line* $l \in Cand_{Poly}$ *where*

$$Cand_{Poly} := \{\text{lines } l\text{: } l \text{ is a facet of } R\}$$
$$\cup \{\text{lines } l\text{: there exist } m_1, m_2 \in \mathcal{M}, V \in ext(R)\text{:}$$
$$V \in l \text{ and } g(l) = w_{m1} d_{ver}(A_{m_1}, l) = w_{m_2} d_{ver}(A_{m_2}, l)\}.$$

4. Generalization to block norms

The main advantage of the vertical distance is, that the unrestricted line location problems are convex. That does not hold any more for block norm distances, even for l_1 the convexity is lost. For block norm distances, however, an easy separation argument helps to solve the problem. In the following two sections we therefore need one more definition, already introduced in [20,22].

Let $t \in \mathbb{R}^2$ be a given direction. For $X \in \mathbb{R}^2$ and a line $l \subset \mathbb{R}^2$ we define the *t-distance* between X and l as

$$d_t(X, l) := \min\{|\lambda|\text{: } X + \lambda t \in l\},$$

where $\min \emptyset := \infty$.

Note that for e_2 is the second unit vector of $\mathbb{R}^2$ we get $d_{e_2} = d_{ver}$. For all other directions $t \neq e_2$ the corresponding location problems $1l/\mathbb{R}^2/R/d_t/\cdot$ can be solved by rotating the existing facilities and the forbidden region (if there is any) such that the problem is transformed to the corresponding problem with vertical distance.

Now, if B is a compact, convex polytope with non-empty interior and extreme points

$$ext(B) = \{b_1, b_2, \dots, b_G, -b_1, -b_2, \dots, -b_G\}, \quad b_i \in \mathbb{R}^2, \ i = 1, \dots, G,$$

we see that $\gamma_B(x) := \min\{|\lambda|\text{: } x \in \lambda B\}$ is a block norm with unit ball B and can be expressed by

$$\gamma_B(X) = \min\left\{\sum_{g=1}^{G} |\lambda_g|\text{: } X = \sum_{g=1}^{G} \lambda_g b_g\right\}.$$

The following lemma has been proved in [20] and is simply based on the fact, that a polygon touches a line in at least one of its extreme points.

Lemma 5. *Let* d_B *be derived from a block norm* γ_B. *Then*

$$d_B(X, l) = \min_{g=1,\dots,G} d_{b_g}(X, l).$$

As a consequence we can solve line location problems with block norm distances by solving the problem for all fundamental directions $b_1, b_2 \dots b_G$ (by transforming these problems to d_{ver} as mentioned above) and then taking the best of these solutions. For a restricted simple polygon R we therefore can generalize the results of Theorems 8 and 11 to block norm distances d_B.

Theorem 12. *Let R be a simple polygon. For the restricted problems $1l/\mathbb{R}^2/R = Polygon/d_B/\cdot$ we have:*

- *Either an optimal line for one of the corresponding unrestricted problems $1l/\mathbb{R}^2/\cdot/d_{b_g}/\cdot$, $g = 1, \ldots, G$, is feasible and optimal for the restricted problem or*
- for the median problem *there exists an optimal line for the restricted problem which is a facet of R or which passes through one of the existing facilities and a vertex of R.*
  for the center problem *there exists an optimal line for the restricted problem which is a facet of R or which is at maximum distance from two of the existing facilities and passes through a vertex of R.*

One thing should be emphasized here. It can happen that the best line for the restricted problem is neither optimal for the unrestricted problem nor a tangent to the restricted set R, i.e. Theorem 7 does not hold for block norm distances, as the following example demonstrates.

We use the set of existing facilities shown in Fig. 1 and the distance function d_γ derived from the following block norm:

$$\gamma(X) = \tfrac{3}{2}|x_1| + |x_2|, \quad X = (x_1, x_2)$$

with extreme points $b_1 = (0,1)$ and $b_2 = (\frac{2}{3}, 0)$. Then the optimal solution l^* for the unrestricted problem $1l/\mathbb{R}^2/\cdot/d_\gamma/\sum$ is the same as for the problem with vertical distance $d_{ver} = d_{b_1}$. With the triangle introduced in Fig. 3 as restriction, l^* is forbidden and one optimal solution l_R^* (also minimizing d_{hor}) is shown in Fig. 6. This line is not optimal for the unrestricted case, nor is a tangent to the restricted set R.

So, in general, it is necessary to determine all optimal solutions of the unrestricted problems for each fundamental direction $d_{b_1}, \ldots, d_{b_G}$ to check if any of these optimal solutions is also feasible in the restricted case. As a consequence one easily can determine the whole set of optimizers for $1l/\mathbb{R}^2/R = Polygon/d_B/\cdot$ That this can be relaxed will be shown in the next section.

5. Generalization to arbitrary norms

According to Minkovsky [14] we define a norm by its unit ball. Let B be a convex, compact set in the plane which contains the origin in its interior and is symmetric with respect to the origin. Then $\gamma_B(x) := \min\{|\lambda| : x \in \lambda B\}$ defines a norm and $d(X, Y) = \gamma(Y - X)$ is the corresponding distance. In the classification scheme a γ in position 4 indicates that we are concerned with an arbitrary norm.

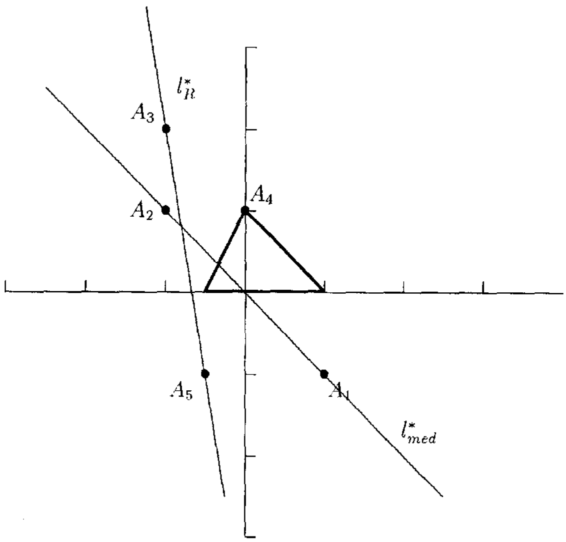

Fig. 6. Optimal solution l_R^* of a restricted problem (with block norm distance) which is neither optimal for the unrestricted case nor a tangent.

Lemma 6. *Let d be a distance derived from a norm γ and let l be a line with slope s. Then there exists $t = t(s) \in \mathbb{R}^2$ (only dependent on the slope s of the line) such that for all $t' \in \mathbb{R}^2$*

$$d(A,l) = d_t(A,l) \leqslant d_{t'}(A,l) \quad \textit{for all } A \in \mathbb{R}^2.$$

Proof. The proof is omitted. It is given in [22]. It uses the fact, that the unit ball B will touch all parallel lines l in the same direction t from the origin to the touching point. □

As any point on the unit ball might touch the optimal line, the only straightforward conclusion is that the optimal line for the restricted problem either is an optimal solution for one of the (infinitely many) problems $1l/\mathbb{R}^2/\cdot/d_t/\cdot$ for all $t \in \mathbb{R}^2$ or the optimal line is a tangent to the restricted set R. Algorithmically this certainly is not very helpful, but with the following lemma (holding for vertical distance d_{ver}) it is possible to derive a finite candidate set also for arbitrary norms.

Lemma 7. *Let R be a connected set and let $\mathscr{L}^*$ be the set of optimal lines for the unrestricted problem with vertical distance $1l/\mathbb{R}^2/\cdot/d_{ver}/\sum$. Moreover suppose that there are lines in $\mathscr{L}^*$ which intersect $int(R)$ and lines in $\mathscr{L}^*$ which are feasible for the restricted problem. Then there exists a line $l \in \mathscr{L}^*$, which is also feasible for the restricted problem $1l/\mathbb{R}^2/R/d_{ver}/\cdot$ and which is a tangent to R and*

• *for the median problem which passes through one of the existing facilities.*
• *for the center problem which is at maximum distance from two of the existing facilities.*

Proof. Both sets $T(\mathscr{L}^*) \cap T(int(R))$ and $T(\mathscr{L}^*) \cap T(\mathbb{R}^2 \setminus int(R))$ are non-empty. As $T(R)$ and $T(\mathscr{L}^*)$ both are connected and have no wholes, their boundaries intersect in a point X, corresponding to a line $l = T^{-1}(X)$. (For this proof let the boundary ∂R of a one-dimensional set R be defined as the set R itself.) As $X \in \partial T(R)$, the line l is a tangent to R according to Theorem 7. Furthermore, any point $Y \in \partial T(\mathscr{L}^*)$ is contained in at least one construction line such that for the median problem there exists $m \in \mathscr{M}$ with $X \in T(A_m)$ which according to Lemma 1 means $A_m \in l$, and for the center problem l is at maximum distance from two of the existing facilities. □

Theorem 13. *Let R be a simple polygon. For the restricted problems* $1l/\mathbb{R}^2/R = Polygon/\gamma/\cdot$ *there exists an optimal line which*
• *for the median problem is a facet of R or which passes through one of the existing facilities and a vertex of R or which passes through two of the existing facilities.*
• *for the center problem is a facet of R or which is at maximum distance from two of the existing facilities and passes through a vertex of R or which is at maximum distance from three of the existing facilities.*

Proof. We prove the result for the median objective function. Let d_γ be the distance derived from γ. Now suppose there is an optimal line l^* for the restricted problem $1l/\mathbb{R}^2/R = Polygon/\gamma/\sum$ that does not fulfill one of the above properties. Choose $t \in \mathbb{R}^2$ such that $d_\gamma(A, l^*) = d_t(A, l^*)$ for all $A \in \mathbb{R}^2$ according to Lemma 6. Consider now the same location problem but with distance d_t instead of d_γ, i.e. $1l/\mathbb{R}^2/R = Polygon/d_t/\sum$. Let us denote by $\mathscr{L}_t^*$ the set of optimal solutions for the unrestricted problem with distance d_t, i.e. for $1l/\mathbb{R}^2/\cdot/d_t/\sum$. Now we choose a line l^0 by considering three cases. Note that Theorem 8 and Lemma 7 hold not only for d_{ver}, but also for d_t by rotation.

– If all lines $l \in \mathscr{L}_t^*$ do intersect $int(R)$ we know from Theorem 8 that there exists a line l^0 which is optimal for the restricted problem $1l/\mathbb{R}^2/R = Polygon/d_t/\sum$ and passes through an existing facility and a vertex of R or which is a facet of R.

– If no line $l \in \mathscr{L}_t^*$ does intersect $int(R)$ then all optimal lines for the unrestricted problem are also feasible for the restricted case. According to Theorem 1 (for distances d_t) we take an optimal line l^0 passing through two of the existing facilities.

– If there exists a line in $\mathscr{L}_t^*$ which intersects $int(R)$ and there also exists a line in $\mathscr{L}_t^*$ which does not intersect $int(R)$ we conclude from Lemma 7 that there exists a feasible line $l^0 \in \mathscr{L}_t^*$ for the restricted problem which passes through one of the existing facilities and a vertex of R.

For the new line l^0 we see from Lemma 6 that $d_\gamma(A,l^0)=\min_{t'\in\mathbb{R}^2} d_{t'}(A,l^0)\leqslant d_t(A,l^0)$ for all $A\in\mathbb{R}^2$. In summary, we estimate the objective value of l^0.

$$\begin{aligned} f(l^*) &= \sum_{m\in\mathcal{M}} w_m d_\gamma(A_m,l^*) \\ &= \sum_{m\in\mathcal{M}} w_m d_t(A_m,l^*) \\ &\geqslant \sum_{m\in\mathcal{M}} w_m d_t(A_m,l^0) \\ &\geqslant \sum_{m\in\mathcal{M}} w_m d_\gamma(A_m,l^0) = f(l^0) \end{aligned}$$

such that l^0 also is optimal for the restricted problem and satisfies one of the above properties. □

6. Extensions

First we give some extensions to other kinds of restrictions which can easily be solved by the theory developed in this paper.

- Suppose we have given two or more simple polygons $R_1,R_2,\ldots,R_K$ through which the line is not allowed to pass. Then we transform all R_k to dual space and get

  $$R^{\mathrm{T}} = T(R_1)\cup T(R_2)\cup\cdots T(R_K)$$

  as a restricted set which consists of one or more connected polygon components. If all optimal solutions $\mathcal{X}^*$ in dual space are forbidden then there exists a connected component R^0 of R^{T} such that $\mathcal{X}^*\subseteq R^0$ (as $\mathcal{X}^*$ is a convex set). This means we can replace R^{T} by R^0 and solve the problem which exactly yields Theorems 12 and 13 (with R is the union of all single forbidden sets R_k.)
- Now consider a polygon F which must be met by the line facility (e.g. a new railway line must pass through some specified region round an industrial area or round a town). Note that this is not the same as a restricted set $R=\mathbb{R}^2\setminus F$. But, again, we look at the dual version of this problem and note that in dual space we have a restricted set $R=\mathbb{R}^2\setminus T(F)$ which consists of two disjoint, convex connected components R_1 and R_2. As before we can replace R either by R_1 or by R_2, and as both sets are convex we get that there exists an optimal line

  for the median problem which passes through one of the existing facilities and a vertex of F or which passes through two of the existing facilities.

  for the center problem which is at maximum distance from two of the existing facilities and passes through a vertex of F or which is at maximum distance from three of the existing facilities.
- As last example we assume that K polygons $F_1,F_2,\ldots,F_K$ must be met by the new facility and finally get again a result as Theorems 12 and 13 with R is the union of the sets F_k.

Another straightforward extension is to allow weights w_m also for the center problem. The same methods can be applied in this case, since the weighted center function also is piecewise linear and convex. Only the cell structure differs from the unweighted case. The extension to more dimensions and the algorithmic implementation of the described procedures in [3] are under research at the moment.

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